Understanding Matrix Multiplication

Pravin K. Verekar, Ph.D.(IISc)

July 15, 2025 and latest revision August 11, 2026

Contents

1 Multiplication of a Matrix and a Column Vector . . 1
2 Column Picture for Linear System $Ax = b$ 1
3 Multiplication of a Row Vector and a Matrix 2
4 Column Picture for $AB = C$ 2
5 Row Picture for $AB = C$ 3
6 Gaussian Elimination using Matrix Algebra 3
7 Elementary Matrices 5
8 Row Exchanges and Permutation Matrices 6

`Sir Isaac Newton: If I have seen further, it is by standing on the shoulders of giants.`

1 Multiplication of a Matrix and a Column Vector

The multiplication of a matrix A and a column vector x, that is Ax, gives a column vector. So, the multiplication Ax can be viewed as a *function* whose output is a column vector. An m by n matrix multiplies an n-dimensional vector and produces an m-dimensional vector.

Let us now consider

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Pravin K. Verekar
La Campala Colony
Miramar, Panaji, Goa
India, Pin Code 403001
E-mail: prof.verekar@gmail.com

The multiplication of matrix A and column vector x gives

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix} \tag{1}$$

Eq. 1 can be rewritten as

$$Ax = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \tag{2}$$

We observe that the product Ax is the linear combination of the columns of A. This we will call: the *column rule*. The multipliers (coefficients) of the columns of A are the appropriate entries (components) of vector x.

2 Column Picture for Linear System $Ax = b$

Consider the following system of linear equations.

$$\begin{aligned} 2x + y + z &= 5 \\ 4x - 6y + 0z &= -2 \\ -2x + 7y + 2z &= 9 \end{aligned}$$

This linear system has an exact solution; the solution is $(x, y, z) = (1, 1, 2)$. The three row equations represent three planes and they intersect in exactly one point which is $(1, 1, 2)$. This is the row picture.

Now we move on to understand the *column picture*. The above linear system can be written as

$$x \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + y \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

The left hand side consists of three column vectors (the three columns of the matrix), whose linear combination

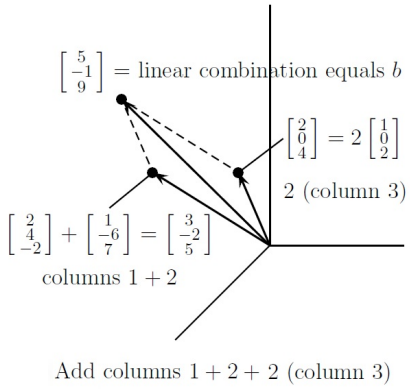

Fig. 1 The figure illustrates the *column picture* where Ax is geometrically represented as the addition of columns of A. If there exists a linear combination of the columns of A that matches vector b, then a unique solution is found for the linear system $Ax = b$. Courtesy Strang (2006).

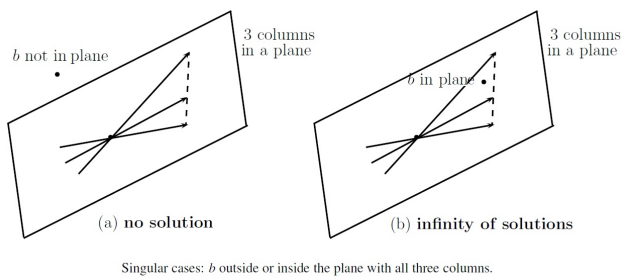

Fig. 2 The figure illustrates the column picture. This figure illustrates singular cases. Courtesy Strang (2006).

with some corresponding multipliers x, y, z, which are to be determined, gives the right hand side vector "b". For this particular example, we know that a unique set of (x, y, z) exists: $(x, y, z) = (1, 1, 2)$. This geometric picture is illustrated in Fig. 1. The addition of columns of A turns out to be the given vector b, and therefore a unique solution exits for the given linear system.

However, yet times there may exist no combination of columns of A that gives vector b; here the linear system has no solution. This is illustrated in Fig. 2 where the three column vectors of the matrix A lie in a plane and end point b lies outside this plane. Other case as shown in Fig. 2 is where point b lies in the plane and there are infinitely many combinations of columns of A that give vector b; the linear system has infinite number of solutions.

When the column vectors of matrix A are *linearly independent*, a unique combination of the columns of A can be found that equals b; in this case the matrix A is non-singular. However whenever few of the columns of A are *linearly dependent*, we find no combination of the columns of A that equals b or infinite number of combinations of the columns of A that equal b; in this case the matrix A is singular.

The determinant of a singular matrix is always zero. The determinant of a nonsingular matrix is never equal to zero.

3 Multiplication of a Row Vector and a Matrix

Consider a matrix A and a vector u as shown

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The multiplication of row vector u^T and matrix A gives

$$u^T A = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$u^T A = \begin{bmatrix} u_1 a_{11} + u_2 a_{21} & u_1 a_{12} + u_2 a_{22} \end{bmatrix} \tag{3}$$

Eq. 3 can be rewritten as

$$u^T A = u_1 \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} + u_2 \begin{bmatrix} a_{21} & a_{22} \end{bmatrix} \tag{4}$$

We observe that the product $u^T A$ is the linear combination of the rows of A. This we will call: the *row rule* or *row picture*. The multipliers (coefficients) of the rows of A are the appropriate entries (components) of vector u^T.

`Remark:` We prefer to use the column or row picture for $n \times n$ matrices, i.e., square matrices.

4 Column Picture for $AB = C$

We now extend the column picture to multiplication of two square matrices A and B. Let the multiplication AB give matrix C. In AB, matrix A is the pre-factor and matrix B is the post-factor.

We shall explain using 3×3 matrices. The multiplication $AB = C$ is shown below.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

In the column picture, a column of C arises from the linear combination of columns of pre-factor matrix A with coefficients taken from the appropriate column of B; the appropriate column number of B chosen is the same as the column number of C.

So, the first column of C is the linear combination of columns of A with coefficients (multipliers) taken from the first column of B, the second column of C is the

linear combination of columns of A with coefficients taken from the second column of B, and so on. The column picture helps us at many places when studying linear algebra. This is depicted below.

$$\underbrace{\begin{matrix} c_{11} \\ c_{21} \\ c_{31} \end{matrix}}_{\text{First column}} = b_{11}\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + b_{21}\begin{bmatrix} a_{12} \\ a_{22} \\ a_{33} \end{bmatrix} + b_{31}\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$$\underbrace{\begin{matrix} c_{12} \\ c_{22} \\ c_{32} \end{matrix}}_{\text{Second column}} = b_{12}\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + b_{22}\begin{bmatrix} a_{12} \\ a_{22} \\ a_{33} \end{bmatrix} + b_{32}\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$$\underbrace{\begin{matrix} c_{13} \\ c_{23} \\ c_{33} \end{matrix}}_{\text{Third column}} = b_{13}\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + b_{23}\begin{bmatrix} a_{12} \\ a_{22} \\ a_{33} \end{bmatrix} + b_{33}\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

5 Row Picture for $AB = C$

We now explain the row picture for multiplication of two square matrices A and B. Let the multiplication AB give matrix C. In AB, the matrix A is the pre-factor and the matrix B is the post-factor.

We shall demonstrate using 2×2 matrices. The multiplication $AB = C$ is shown below.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

In the row picture, a row of C is obtained from the linear combination of rows of post-factor matrix B with coefficients taken from the appropriate row of A; the appropriate row number of A selected is the same as the row number of C.

So, the first row of C is the linear combination of rows of B with coefficients (multipliers) taken from the first row of A, the second row of C is the linear combination of rows of B with coefficients taken from the second row of A, and so on. The row picture is used at many places when studying linear algebra especially *when deciding elementary matrices for Gaussian elimination.* The row picture for our example is shown below.

$$\underbrace{c_{11}\ c_{12}}_{\text{First row}} = a_{11}\begin{bmatrix} b_{11} & b_{12} \end{bmatrix} + a_{12}\begin{bmatrix} b_{21} & b_{22} \end{bmatrix}$$

$$\underbrace{c_{21}\ c_{22}}_{\text{Second row}} = a_{21}\begin{bmatrix} b_{11} & b_{12} \end{bmatrix} + a_{22}\begin{bmatrix} b_{21} & b_{22} \end{bmatrix}$$

`Remarks:`
1. The *row picture*: Each row of AB is a combination of the rows of B.
2. The *column picture*: Each column of AB is a combination of the columns of A.

6 Gaussian Elimination using Matrix Algebra

With our knowledge of the row picture, we can now turn our attention to using matrix algebra for Gaussian elimination. We shall illustrate with an example.

Example 1 : Solve the following three linear equations with three unknowns using Gaussian elimination. Show each step by using matrix algebra (shown in shaded boxes).

$$\begin{aligned} L_1 &: \quad x + y + z = 5 \\ L_2 &: \quad x + 2y + 3z = 7 \\ L_3 &: \quad x + 3y + 6z = 11 \end{aligned} \tag{5}$$

The linear system in Eq. 5 is written using matrices in the form $Ax = b$ where A is coefficient matrix, x is column vector of unknowns and b is column vector of constants.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix} \tag{6}$$

At first, we eliminate x from L_2 and L_3. We replace L_2 by L_2' with $L_2' = -L_1 + L_2$ and replace L_3 by L_3' with $L_3' = -L_1 + L_3$. These *elimination* operations, also called *elementary* operations yield the following *row-equivalent* system.

$$\begin{aligned} L_1 &: \quad x + y + z = 5 \\ L_2' &: \quad y + 2z = 2 \\ L_3' &: \quad 2y + 5z = 6 \end{aligned} \tag{7}$$

Using our understanding of the row picture for matrix multiplication, we create a pre-factor matrix E_1 to be multiplied with the coefficient matrix A, remembering that the pre-factor matrix E_1 supplies the multipliers for the row combinations of A. Let the rows of A be named R_1, R_2 and R_3. The matrix E_1 when multiplied with A should not change the first row R_1 in the product matrix, the second row of the product matrix should be $-R_1 + R_2$ and the third row of the product matrix

should be $-R_1 + R_3$. This matrix E_1, which imposes elimination operations on A, is called the *elimination matrix* or *elementary matrix*.

Using elementary matrix E_1, we have $E_1Ax = E_1b$ in Eq. 8.

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix} \tag{9}$$

We write Eq. 9, in short as $A'x = b'$, which is same as Eq. 7.

Then, we eliminate y from L_3'. We replace L_3' by L_3'' with $L_3'' = -2L_2' + L_3'$. The original system gets changed to equivalent *upper triangular* form.

$$\begin{aligned} L_1 : \quad x + y + z &= 5 \\ L_2' : \quad y + 2z &= 2 \\ L_3'' : \quad z &= 2 \end{aligned} \tag{10}$$

On *back-substitution*, we get $z = 2$, $y = -2$ and $x = 5$.

Next, we create a pre-factor matrix E_2 to be multiplied with the coefficient matrix A', remembering that the pre-factor matrix E_2 supplies the multipliers for the row combinations of A'. Let the rows of A' be named R_1', R_2' and R_3'. The matrix E_2 when multiplied with A' should not change the first row R_1' in the product matrix, the second row of the product matrix should also remain R_2' and the third row of the product matrix should be $-2R_2' + R_3'$.

Using elementary matrix E_2, we have $E_2A'x = E_2b'$ in Eq. 11.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} \tag{12}$$

We write Eq. 12, in short as $A''x = b''$, which is same as Eq. 10.

`Remark:` Using elementary row operations, we have obtained *row equivalent matrices* A, A' and A''. This can be written mathematically as $A \sim A' \sim A''$ or as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix A is reduced to *upper triangular matrix* A'', which we will rename as U. The sequence of elimination matrices E_1 and E_2, in that order, changed A to U.

$$E_2E_1A = U$$

Elimination matrices, also called elementary matrices, are square matrices. We can combine elementary matrices, maintaining order of multiplication; so, say, $E = E_2E_1$.

$$EA = U$$

How can we get back A from U? How can we undo the steps of Gaussian elimination?
Elementary matrices execute row operations on matrix multiplication. The matrices E_1 and E_2 are listed below.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Subtraction can be undone by addition; multiplication can be undone by division. One operation cancels the other. To get back A from U, the operations of the elementary matrices are to be *inverted*. So, what we get are inverse matrices E_1^{-1} and E_2^{-1}.

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Note that the inverse of elementary matrices can be obtained by inspection.

Let us complete the steps from U to A. The matrix equation from Gaussian elimination is

$$EA = U, \qquad E_2E_1A = U$$

We invert the steps to get A.

$$E^{-1}EA = E^{-1}U, \qquad (E_2E_1)^{-1}E_2E_1A = (E_2E_1)^{-1}U$$

$$E_1^{-1}E_2^{-1}E_2E_1A = E_1^{-1}E_2^{-1}U, \qquad IA = E_1^{-1}E_2^{-1}U$$

$$A = E_1^{-1}E_2^{-1}U$$

We do the multiplication $E_1^{-1}E_2^{-1}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

The multiplication $E_1^{-1}E_2^{-1}$ yields *lower triangular matrix*, with 1s on the diagonal. The multipliers that take backwards from U to A are all below the diagonal. We name the lower triangular matrix $E_1^{-1}E_2^{-1}$ as L.

The coefficient matrix A of a system of linear equations having an exact solution is always square and invertible (that is, nonsingular), as in *Example 1*. Such a matrix A can always be reduced to an upper triangular form U using Gaussian elimination, that is, multiplying A by elementary matrix E. The matrix E is also nonsingular.

Whenever the elementary operation of row exchange in not needed during Gaussian elimination, the inverse of E has the lower triangular form L with 1s on the diagonal and we have $LU = A$. This is the classical *LU factorization*, also called *LU decomposition*. The equivalent matrix U of the coefficient matrix A has *pivots* in all columns.

We make following important remarks for nonsingular matrices.

1. Any nonsingular matrix A can be reduced to upper triangular form U using elementary row operations.
2. The diagonal entries of U are pivots; it means that there will be no 0 in the diagonal.
3. Obtaining an LU decomposition for a nonsingular matrix A requires that the steps in Gaussian elimination proceed without needing interchange of rows.
4. The determinant of A is the same as the determinant of U and it is the product of all the entries of the diagonal of U.

7 Elementary Matrices

Consider a matrix A with rows $R_1, R_2, \ldots, R_m$. We can carry out the following *elementary row operations* on this matrix A.

1. (Row exchange): Interchange of rows R_i and R_j.
2. (Row scaling): Replace row R_i by a nonzero multiple kR_i.
3. (Row addition): Replace row R_j by the sum of a multiple kR_i and itself (i.e. R_j).

Corresponding to the respective elementary row operations, we can set up appropriate *elementary matrices*. The following example illustrate this.

Example 2 : Construct 3×3 elementary matrices corresponding to the following elementary row operations: (a) Interchange of R_2 and R_3, (b) Replace R_2 by $-6R_2$, (c) Replace R_3 by $-4R_1 + R_3$.

Elementary matrices are easily conceived from our understanding of the row picture for matrices and they are as given below.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

Note that elementary matrices are square matrices, and they are invertible. This means that any product $E = E_k \ldots E_2 E_1$ is also invertible.

`Remarks:`
Each elementary matrix carries out separate elimination steps (row scaling or row addition). We also know that the identity matrix does nothing at all ($IA = A$). We can view elementary matrix as obtained from the identity matrix on execution of the row operation on it.

7.1 Elementary Column Operations

Just as elementary row operations, similarly we have *elementary column operations*. Consider a matrix A with columns $C_1, C_2, \ldots, C_n$. Elementary column operations on A are listed below.

1. (Column exchange): Interchange of columns C_i and C_j.
2. (Column scaling): Replace column C_i by a nonzero multiple kC_i.
3. (Column addition): Replace column C_j by the sum of a multiple kC_i and itself (i.e. C_j).

Each column operation has an inverse operation.

We have *elementary column matrix* corresponding to an elementary column operation; we shall name it F. From our understanding of the column picture, the elementary column matrix F will be a postfactor to A, that is, AF. Note that F is always a square matrix and has an inverse.

Example 3 : Construct 3×3 elementary matrices corresponding to the following elementary column operations:

(a) Interchange of C_1 and C_3, (b) Replace C_3 by $-2C_3$, (c) Replace C_3 by $-3C_2 + C_3$.

Elementary matrices are easily conceived from our understanding of the column picture for matrices and they are as given below.

$$F_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

`Remarks:`
It is interesting and important to note that any elementary operation involving the rows of a matrix A can be accomplished by premultiplying A by a unit matrix on whose rows the same elementary operation has been performed, and any elementary operation involving the columns of a matrix A can be accomplished by postmultiplying A by a unit matrix on whose columns the same elementary operation has been performed. The matrix A can be any arbitrary $m \times n$ matrix.

Any nonsingular $n \times n$ matrix can be obtained from the identity matrix I_n by a sequence of elementary row operations or a sequence of elementary column operations.

8 Row Exchanges and Permutation Matrices

Elementary matrix that does row exchange is called *permutation matrix*. Permutation matrix P has only 0s and 1s, with exactly one 1 in each row and column. We can view permutation matrix as obtained from the identity matrix on execution of the row exchange on it. The most common permutation matrix is $P = I$ (it exchanges nothing). The product of two permutation matrices is another permutation— the rows of I get reordered twice. If row exchanges or permutations P_1, P_2, ..., P_k, in that order, are used, then the permutation can be combined into one matrix $P = P_k \cdots P_2 P_1$. Since permutation matrices are a subset of elementary matrices, whatever we stated earlier for elementary matrices apply to permutation matrices; for example, permutation matrices are square matrices, and they are invertible (that is, the operations of row exchanges can be reversed).

`Remarks:`
Permutation matrices can be for both row exchanges or column exchanges. Permutation matrices are orthogonal, meaning their inverse is equal to their transpose ($P^{-1} = P^T$). Their determinant is either +1 or -1, depending on the permutation.

In Gaussian elimination, two equations may need to be interchanged in order to obtain a nonzero coefficient as pivot. In matrix algebra, this is accomplished by multiplying a permutation matrix with the coefficient or augmented matrix.

Strang (2006): In practice, we also consider a row exchange when the original pivot is *near* zero— even if it is not exactly zero. Choosing a larger pivot reduces the roundoff error.

`Partial pivoting:` During Gaussian reduction to echelon form, the value for the current pivot position in a row is chosen to be the largest absolute value in the current pivot column from the rows below the current row and then interchange the current row with the relevant row. This process is done starting with the first pivot to the next pivot, and onwards.

`Total or complete or full pivoting:` In complete pivoting, the largest absolute value is obtained in the current pivot position by either interchanging rows or interchanging columns. However, interchange of columns involves change in positions of the variables in the solution vector, and this has to be kept track of whenever column swapping is done.

LU `decomposition for` A `with elimination involving row exchanges (`$PA = LU$`):`
Whenever Gaussian elimination needs row exchanges for a nonsingular matrix A in order to have all the pivots on the diagonal in the reduced matrix, then those exchanges are done first (by P). The matrix PA now will not need any further row exchanges. So, with the rows reordered in advance, PA can be factored into LU.

References

Kreyszig, E., Kreyszig, H. & Norminton, E. J. 2011 *Advanced Engineering Mathematics*, tenth edn. John Wiley & Sons.

Lipschutz, S. & Lipson, M. L. 2001 *Schaum's Outlines: Linear Algebra*, 3rd edn. McGraw-Hill.

Strang, G. 2006 *Linear Algebra and Its Applications*, 4th edn. Cengage Learning.

Wylie, C. R. 1966 *Advanced Engineering Mathematics*, 3rd edn. McGraw-Hill.